Seismic pulse propagation with constant Q and stable probability distributions

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This note is dedicated to Professor Michele Caputo in occasion of his 70-th birthday. Throughout his intensive and outstanding career Professor Caputo has recognized the importance of the quality factor Q and fractional calculus in seismology, providing interesting contributions on these topics.

Abstract

The one-dimensional propagation of seismic waves with constant Q is shown to be governed by an evolution equation of fractional order in time, which interpolates the heat equation and the wave equation. The fundamental solutions for the Cauchy and Signalling problems are expressed in terms of entire functions (of Wright type) in the similarity variable and their behaviours turn out to be intermediate between those for the limiting cases of a perfectly viscous fluid and a perfectly elastic solid. In view of the small dissipation exhibited by the seismic pulses, the nearly elastic limit is considered. Furthermore, the fundamental solutions for the Cauchy and Signalling problems are shown to be related to stable probability distributions with index of stability determined by the order of the fractional time derivative in the evolution equation.

Key words: *Earth anelasticity - Quality factor - Wave propagation - Fractional derivatives - Stable probability distributions*

1 Introduction

In seismology the problem of wave attenuation due to anelasticity of the Earth is described by the so-called quality factor Q, or, better, by its inverse Q^{-1} (internal friction or loss tangent), which is related to the dissipation of the elastic energy during the wave propagation. Because of its great relevance in determining the composition and the mechanical properties of the Earth, the problem has been considered from different points of view by many researchers. Without pretending to be exhaustive, we quote (in alphabetic order of the first author) some original contributions and reviews among those which have attracted our attention, *e.g.* Aki and Richards (1980), Ben-Menhahem and Singh (1981), Caputo (1966, 1967, 1969, 1976, 1979, 1981, 1985, 1996a) Caputo and Mainardi (1971), Carcione *et al.* (1988), Chin (1980), Futterman (1962), Gordon and Nelson (1966), Jackson and Anderson (1970), Kanamori and Anderson (1977), Kang and McMechan (1993) Kjartansson (1979), Knopoff (1964), Körnig and Müller (1989), Mitchell (1995), Murphy (1982), O'Connel and Budiansky (1978), Ranalli (1987), Sabadini *et al.* (1985, 1987), Savage and O'Neill (1975), Spencer (1981), Strick (1967, 1970, 1982, 1984), Strick and Mainardi (1982), Yuen *et al.* (1986).

It is known that seismic pulse propagation mostly occurs with a quality factor Q constant over a wide range of frequencies. As pointed out by Caputo and Mainardi (1971) and Caputo (1976), this factor turns out to be independent of frequency only in special linear viscoelastic media for which the stress is proportional to a fractional derivative of the strain, of order ν less than one. Since these media exhibit a creep compliance depending on time by a power-law with exponent ν, we refer to them as power-law solids, according to the notation by Kolsky (1956) and Pipkin (1972-1986).

For the sake of convenience, the generalized operators of integration and differentiation of arbitrary order are recalled in the Appendix in the framework of the so-called Riemann-Liouville *Fractional Calculus*. In this paper we adopt the Caputo definition for the *fractional derivative of order* $\alpha > 0$ of a causal function $f(t)$ (*i.e.* vanishing for $t < 0$),

$$\frac{d^\alpha}{dt^\alpha}\, f(t) := \begin{cases} f^{(m)}(t) & \text{if}\quad \alpha = m \in \mathbb{N}\,, \\ \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau & \text{if}\quad m-1 < \alpha < m\,, \end{cases} \tag{1.1}$$

where $f^{(m)}(t)$ denotes the derivative of integer order m and Γ is the Gamma function.

In Section 2 we derive the general evolution equation governing the propagation of uniaxial stress waves, in the framework of the dynamical theory of linear viscoelasticity. For a power-law solid the evolution equation is shown to be of fractional order in time, which is intermediate between the heat equation and the wave equation. In fact, denoting the space and time variables by x and t and the response field variable by $w(x,t)$, the evolution equation will be shown to be

$$\frac{\partial^{2\beta} w}{\partial t^{2\beta}} = D\, \frac{\partial^2 w}{\partial x^2}\,, \qquad 2\beta = 2 - \nu\,. \tag{1.2}$$

The order of the time derivative has been denoted by 2β for reasons that will appear later. Since $0 < \nu \leq 1$, we get $1/2 \leq \beta < 1$.

In Section 3 we review the analysis of the fractional evolution equation (1.2) in the general case $0 < \beta < 1$, essentially based on our works, Mainardi (1994, 1995, 1996a, 1996b). We first analyse the two basic boundary-value problems, referred to as the *Cauchy* problem and the *Signalling* problem, by the technique of the Laplace transform and we derive the transformed expressions of the respective fundamental solutions (the Green functions). Then, we carry out the inversion of the relevant Laplace transforms and we outline a *reciprocity relation* between the Green functions in the space-time domain. In view of this relation the Green functions can be expressed in terms of two interrelated *auxiliary functions* in the similarity variable $r = |x|/(\sqrt{D}t^{\beta})$. These auxiliary functions can be analytically continued in the whole complex plane as entire functions of Wright type.

In Section 4 we show the evolution of the fundamental solutions for $1/2 \leq \beta < 1$, that can be relevant in seismology to simulate the propagation of seismic pulses. Accounting for the low dissipation occurring in the Earth, the nearly elastic limit must be considered; in this case the pulse response becomes a narrow, sharply peaked function and the arguments by Pipkin (1972-1986) and Kreis and Pipkin (1986) must be adopted in order to obtain an evaluation of the solutions, which is suitable from numerical point of view.

Finally, in Section 5, following Kreis and Pipkin (1986), we point out the interesting connection between the fundamental solution for the *Signalling* problem and the density of a certain (unilateral) *stable* probability distribution. We note that this connection generalizes the one known for the standard heat equation for which the fundamental solution for the *Signalling* problem is related to the density of the *stable Lévy* distribution. Since the above property is expected to provide a further insight into our evolution equation of fractional order, the seismic pulse propagation with constant Q assumes an additional interest from a mathematical-physical point of view.

2　Linear Viscoelastic Waves and the Fractional Diffusion-Wave Equation

According to the elementary one-dimensional theory of linear viscoelasticity, the medium is assumed to be homogeneous (of density ρ), semi-infinite or infinite in extent ($0 \leq x < +\infty$ or $-\infty < x < +\infty$) and undisturbed for $t < 0$. The basic equations are known to be, see *e.g.* Hunter (1960), Caputo & Mainardi (1971), Pipkin (1972-1986), Christensen (1972-1982), Chin (1980), Graffi (1982),

$$\sigma_x(x,t) = \rho\, u_{tt}(x,t)\,, \tag{2.1}$$

$$\epsilon(x,t) = u_x(x,t)\,, \tag{2.2}$$

$$\epsilon(x,t) = [J_0 + \dot{J}(t)*]\,\sigma(x,t)\,. \tag{2.3}$$

Here the suffices x and t denote partial derivation with respect to space and time respectively, the dot ordinary time-derivation, and the star integral time-convolution from 0^+ to t. The following notations have been used: σ for the

stress, ϵ for the strain, $J(t)$ for the creep compliance (the strain response to a unit step input of stress); the constant $J_0 := J(0^+) \geq 0$ denotes the instantaneous (or glass) compliance.

The evolution equation for the *response variable* $w(x, t)$ (chosen among the field variables: the displacement u, the stress σ, the strain ϵ or the particle velocity $v = u_t$) can be derived through the application of the Laplace transform to the basic equations. We use the following notation for the Laplace transform of a function $f(t)$, locally summable for $t \geq 0$,

$$\mathcal{L}\{f(t)\} := \int_0^\infty \mathrm{e}^{-st} f(t)\, dt = \widetilde{f}(s)\,, \ s \in \mathbb{C}\,,$$

and we adopt the sign $\div$ to denote a Laplace transform pair, *i.e.* $f(t) \div \widetilde{f}(s)$.

We first obtain in the transform domain, the second order differential equation

$$\left[\frac{d^2}{dx^2} - \mu^2(s) \right] \widetilde{w}(x, s) = 0\,, \tag{2.4}$$

in which

$$\mu(s) := s \left[\rho\, s \widetilde{J}(s) \right]^{1/2} \tag{2.5}$$

is real and positive for s real and positive. As a matter of fact, $\mu(s)$ turns out to be an analytic function of s over the entire s-plane cut along the negative real axis; the cut can be limited or unlimited in accordance with the particular viscoelastic model assumed.

Wave like or diffusion like character of the evolution equation can be drawn from (2.5) by taking into account the asymptotic representation of the creep compliance for short times,

$$J(t) = J_0 + O(t^\nu)\,, \quad \text{as } t \to 0^+\,, \tag{2.6}$$

with $J_0 \geq 0$, and $0 < \nu \leq 1$. If $J_0 > 0$ then

$$\lim_{s \to \infty} \frac{\mu(s)}{s} = \sqrt{\rho J_0} := \frac{1}{c}\,, \tag{2.7}$$

we have a wave like behaviour with c as the wave-front velocity; otherwise ($J_0 = 0$) we have a diffusion like behaviour. In the case $J_0 > 0$ the wave like evolution equation for $w(x, t)$ can be derived by inverting (2.4-5), using (2.6-7) and introducing the non dimensional rate of creep

$$\psi(t) := \frac{1}{J_0} \frac{dJ(t)}{dt} \geq 0\,, \quad t > 0\,. \tag{2.8}$$

We get

$$\mu^2(s) := s^2 [\rho\, s \widetilde{J}(s)] = \left(\frac{s}{c} \right)^2 [1 + \widetilde{\psi}(s)]\,, \tag{2.9}$$

so that the evolution equation turns out to be

$$\{1 + \psi(t) * \} \frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}\,. \tag{2.10}$$

This is a generalization of D'Alembert wave equation in that it is an integro-differential equation where the convolution integral can be interpreted as a perturbation term. This case has been investigated by Buchen and Mainardi (1975) and by Mainardi and Turchetti (1975), who have provided wave-front expansions for the solutions.

In the case $J_0 = 0$ we can re-write (2.6) as

$$J(t) = \frac{1}{\rho D} \frac{t^\nu}{\Gamma(\nu+1)} + o\left(t^\nu\right), \quad \text{as } t \to 0^+, \tag{2.11}$$

where, for the sake of convenience, we have introduced the positive constant D (whose dimensions are $L^2 T^{\nu-2}$) and the Gamma function $\Gamma(\nu+1)$. Then we can introduce the non-dimensional function $\phi(t)$ whose Laplace transform is such that

$$\mu^2(s) := s^2 \left[\rho\, s\widetilde{J}(s)\right] = \frac{s^{2-\nu}}{D} \left[1 + \widetilde{\phi}(s)\right]. \tag{2.12}$$

Using (2.12), the Laplace inversion of (2.4-5) yields

$$\{1 + \phi(t) *\} \frac{\partial^{2\beta} w}{\partial t^{2\beta}} = D \frac{\partial^2 w}{\partial x^2}, \tag{2.13}$$

where $2\beta = 2 - \nu$ so $1/2 \le \beta < 1$. Here the time-derivative turns out to be just the fractional derivative of order 2β (in Caputo's sense), according to the Riemann-Liouville theory of *Fractional Calculus* recalled in the Appendix.

When the creep compliance satisfies the simple power-law

$$J(t) = \frac{1}{\rho D} \frac{t^\nu}{\Gamma(\nu+1)}, \quad t > 0, \tag{2.14}$$

we obtain $\phi(t) \equiv 0$, and the evolution equation (2.13) simply reduces to (1.2). As pointed out by Caputo and Mainardi (1971), the creep law (2.14) is provided by viscoelastic models whose stress-strain relation (2.3) can be simply expressed by a fractional derivative of order ν. In the present notation this stress-strain relation reads

$$\sigma = \rho D \frac{d^\nu}{dt^\nu} \epsilon, \quad 0 < \nu \le 1. \tag{2.15}$$

For $\nu = 1$ the Newton law for a viscous fluid is recovered from (2.15) where D now represents the kinematic viscosity; in this case, since $\beta = 1/2$ in (1.2), the classical diffusion equation (or heat equation) holds for $w(x,t)$. When $0 < \nu < 1$ the evolution equation (1.2) turns out to be intermediate between the heat equation and the wave equation. In general we refer to (1.2) as the fractional diffusion-wave equation, and its solutions can be interpreted as fractional diffusive waves, see Mainardi (1995).

We point out that the viscoelastic models based on (2.14) or (2.15) with $0 < \nu < 1$ and henceforth governed by an evolution equation of fractional order in time, see (1.2) with $1/2 < \beta < 1$, are of great interest in material sciences and seismology. In fact, as shown by Caputo and Mainardi (1971), these models exhibit an internal friction independent on frequency according to the law

$$Q^{-1} = \tan\left(\frac{\nu\pi}{2}\right) \iff \nu = \frac{2}{\pi} \arctan\left(Q^{-1}\right). \tag{2.16}$$

The independence of the Q from the frequency is in fact experimentally verified in pulse propagation phenomena for many materials including those of seismological interest. From (2.16) we note that the Q is also independent on the material constants ρ and D which, however, play a role in the phenomenon of wave dispersion.

The limiting cases of absence of energy dissipation (the elastic energy is fully stored) and of absence of energy storage (the elastic energy is fully dissipated) are recovered from (2.16) for $\nu = 0$ (perfectly elastic solid) and $\nu = 1$ (perfectly viscous fluid), respectively.

To obtain values of seismological interest for the dissipation ($Q \approx 1000$) we need to choose the parameter ν sufficiently close to zero, which corresponds to a *nearly elastic* material; from (2.16) we obtain the approximate relations between ν and Q, namely

$$\nu \approx \left(\frac{2}{\pi Q} \right) \approx 0.64 \, Q^{-1} \iff Q^{-1} \approx \frac{\pi}{2} \, \nu \approx 1.57 \, \nu \,. \qquad (2.17)$$

As a matter of fact the evolution equation (1.2) turns out to be a linear Volterra integro-differential equation of convolution type with a weakly singular kernel of Abel type. Equations of this kind have been treated, both with and without reference to the fractional calculus, by a number of authors including Caputo (1969, 1976, 1996b), Meshkov and Rossikhin (1970), Pipkin (1972-1986), Buchen and Mainardi (1975), Kreis and Pipkin (1986), Nigmatullin (1986), Schneider and Wyss (1989), Giona and Roman (1992), Metzler *et al.* (1994) and Mainardi (1994, 1995, 1996a, 1996b). For recent reviews on related topics we refer to Rossikhin and Shitikova (1997) and Mainardi (1997).

3 The Reciprocity Relation and the Auxiliary Functions

The two basic problems for our fractional wave equation (1.2) concern, for $t \geq 0$, the infinite interval $-\infty < x < +\infty$ and the semi-infinite interval $x \geq 0$, respectively; the former is an initial - value problem, referred to as the *Cauchy* problem, the latter is an initial boundary - value problem, referred to as the *Signalling* problem.

Extending the classical analysis to our fractional equation (1.2), and denoting by $g(x)$ and $h(t)$ two given, sufficiently well-behaving functions, the basic problems are thus formulated as following,

a) *Cauchy* problem,

$$w(x, 0^+) = g(x)\,, \quad -\infty < x < +\infty\,; \qquad w(\mp\infty, t) = 0\,, \quad t > 0\,; \qquad (3.1a)$$

b) *Signalling* problem,

$$w(x, 0^+) = 0\,, \quad x > 0\,; \qquad w(0^+, t) = h(t)\,, \quad w(+\infty, t) = 0\,, \quad t > 0\,. \quad (3.1b)$$

If $1/2 < \beta < 1$, we must add in (3.1a) and (3.1b) the initial values of the first time derivative of the field variable, $w_t(x, 0^+)$, since in this case (1.2) contains a time derivative of the second order. To ensure the continuous dependence of our solution with respect to the parameter β also in the transition from $\beta = (1/2)^-$ to $\beta = (1/2)^+$, we agree to assume $w_t(x, 0^+) = 0$.

In view of our analysis we find it convenient from now on to add the parameter β to the independent space-time variables x, t in the solutions, writing $w = w(x, t; \beta)$.

For the *Cauchy* and *Signalling* problems we introduce the so-called Green functions $\mathcal{G}_c(x, t; \beta)$ and $\mathcal{G}_s(x, t; \beta)$, which represent the respective fundamental solutions, obtained when $g(x) = \delta(x)$ and $h(t) = \delta(t)$. As a consequence, the solutions of the two basic problems are obtained by a space or time convolution according to

$$w(x, t; \beta) = \int_{-\infty}^{+\infty} \mathcal{G}_c(x - \xi, t; \beta) \, g(\xi) \, d\xi \,, \qquad (3.2a)$$

$$w(x, t; \beta) = \int_{0^-}^{t^+} \mathcal{G}_s(x, t - \tau; \beta) \, h(\tau) \, d\tau \,. \qquad (3.2b)$$

It should be noted that in (3.2a) $\mathcal{G}_c(x, t; \beta) = \mathcal{G}_c(|x|, t; \beta)$ since the Green function of the Cauchy problem turns out to be an even function of x. According to a usual convention, in (3.2b) the limits of integration are extended to take into account for the possibility of impulse functions centred at the extremes.

For the standard diffusion equation ($\beta = 1/2$) it is well known that

$$\mathcal{G}_c(x, t; 1/2) := \mathcal{G}_c^d(x, t) = \frac{1}{2\sqrt{\pi D}} \, t^{-1/2} \, \mathrm{e}^{-x^2/(4\,D\,t)} \,, \qquad (3.3a)$$

$$\mathcal{G}_s(x, t; 1/2) := \mathcal{G}_s^d(x, t) = \frac{x}{2\sqrt{\pi D}} \, t^{-3/2} \, \mathrm{e}^{-x^2/(4\,D\,t)} \,. \qquad (3.3b)$$

In the limiting case $\beta = 1$ we recover the standard wave equation, for which, putting $c = \sqrt{D}$,

$$\mathcal{G}_c(x, t; 1) := \mathcal{G}_c^w(x, t) = \frac{1}{2} \left[\delta(x - ct) + \delta(x + ct) \right] \,, \qquad (3.4a)$$

$$\mathcal{G}_s(x, t; 1) := \mathcal{G}_s^w(x, t) = \delta(t - x/c) \,. \qquad (3.4b)$$

In the general case $0 < \beta < 1$ the two Green functions will be determined by using the technique of the Laplace transform. This technique allows us to obtain the transformed functions $\widetilde{\mathcal{G}_c}(x, s; \beta)$, $\widetilde{\mathcal{G}_s}(x, s; \beta)$, by solving ordinary differential equations of the 2-nd order in x and then, by inversion, $\mathcal{G}_c(x, t; \beta)$ and $\mathcal{G}_s(x, t; \beta)$.

For the *Cauchy* problem (3.1a) the application of the Laplace transform to (1.2) with $w(x, t) = \mathcal{G}_c(x, t; \beta)$ leads to the non homogeneous differential equation satisfied by the image of the Green function, $\widetilde{\mathcal{G}_c}(x, s; \beta)$,

$$D \frac{d^2 \widetilde{\mathcal{G}_c}}{dx^2} - s^{2\beta} \, \widetilde{\mathcal{G}_c} = -\delta(x) \, s^{2\beta - 1} \,, \quad -\infty < x < +\infty \,. \qquad (3.5)$$

Because of the singular term $\delta(x)$ we have to consider the above equation separately in the two intervals $x < 0$ and $x > 0$, imposing the boundary conditions at $x = \mp\infty$, $\mathcal{G}_c(\mp\infty, t; \beta) = 0$, and the necessary matching conditions at $x = 0^\pm$. We obtain

$$\widetilde{\mathcal{G}_c}(x, s; \beta) = \frac{1}{2\sqrt{D}\, s^{1-\beta}}\, e^{-(|x|/\sqrt{D})s^\beta}, \quad -\infty < x < +\infty. \qquad (3.6)$$

For the *Signalling* problem (3.1b) the application of the Laplace transform to (1.2) with $w(x, t) = \mathcal{G}_s(x, t; \beta)$ leads to the homogeneous differential equation

$$D\, \frac{d^2 \widetilde{\mathcal{G}_s}}{dx^2} - s^{2\beta}\, \widetilde{\mathcal{G}_s} = 0, \quad x \geq 0. \qquad (3.7)$$

Imposing the boundary conditions at $x = 0$, $\mathcal{G}_s(0^+, t; \beta) = h(t) = \delta(t)$, and at $x = +\infty$, $\mathcal{G}_s(+\infty, t; \beta) = 0$, we obtain

$$\widetilde{\mathcal{G}_s}(x, s; \beta) = e^{-(x/\sqrt{D})s^\beta}, \quad x \geq 0. \qquad (3.8)$$

From (3.6) and (3.8) we recognize for the original Green functions the following *reciprocity relation*

$$2\beta\, x\, \mathcal{G}_c(x, t; \beta) = t\, \mathcal{G}_s(x, t; \beta), \quad x > 0, \quad t > 0. \qquad (3.9)$$

This relation can be easily verified in the case of standard diffusion ($\beta = 1/2$), where the explicit expressions (3.4) of the Green functions leads to the identity

$$x\, \mathcal{G}_c^d(x, t) = t\, \mathcal{G}_s^d(x, t) = \frac{1}{2\sqrt{\pi}}\, \frac{x}{\sqrt{D\, t}}\, e^{-x^2/(4\, D\, t)} = F^d(r) = \frac{r}{2}\, M^d(r), \quad (3.10)$$

where $r = x/(\sqrt{D}\, t^{1/2}) > 0$ is the well-known *similarity variable* and

$$M^d(r) = \frac{1}{\sqrt{\pi}}\, e^{-r^2/4}. \qquad (3.11)$$

We refer to $F^d(r)$ and $M^d(r)$ as to the *auxiliary functions* for the diffusion equation because each of them provides the fundamental solution through (3.10). We note that $M^d(r)$ satisfies the normalization condition $\int_0^\infty M^d(r)\, dr = 1$.

Applying in the reciprocity relation (3.9) the complex inversion formula for the transformed Green functions (3.6) and (3.8), and changing the integration variable in $\sigma = s\, t$, we obtain

$$2\beta\, x\, \mathcal{G}_c(x, t; \beta) = t\, \mathcal{G}_s(x, t; \beta) = F(r; \beta) = \beta r\, M(r; \beta). \qquad (3.12)$$

where

$$r = x/(\sqrt{D}\, t^\beta) > 0 \qquad (3.13)$$

is the *similarity variable* and

$$F(r; \beta) := \frac{1}{2\pi i} \int_{Br} e^{\sigma\, -\, r\sigma^\beta}\, d\sigma, \quad M(r; \beta) := \frac{1}{2\pi i} \int_{Br} e^{\sigma\, -\, r\sigma^\beta}\, \frac{d\sigma}{\sigma^{1-\beta}} \qquad (3.14)$$

are the *auxiliary functions*. In (3.14) Br denotes the Bromwich path and $r > 0$, $0 < \beta < 1$.

The above definitions of $F(r;\beta)$ and $M(r;\beta)$ by the Bromwich representation can be analytically continued from $r > 0$ to any $z \in \mathbb{C}$, by deforming the Bromwich path Br into the Hankel path Ha, a contour that begins at $\sigma = -\infty - ia$ $(a > 0)$, encircles the branch cut that lies along the negative real axis, and ends up at $\sigma = -\infty + ib$ $(b > 0)$.

The integral and series representations of $F(z;\beta)$ and $M(z;\beta)$, valid on all of $\mathbb{C}$, with $0 < \beta < 1$ turn out to be

$$
\begin{aligned}
F(z;\beta) &= \frac{1}{2\pi i} \int_{Ha} e^{\sigma - z\sigma^\beta} \, d\sigma \\
&= \sum_{n=1}^{\infty} \frac{(-z)^n}{n! \, \Gamma(-\beta n)} \\
&= -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^n}{n!} \, \Gamma(\beta n + 1) \, \sin(\pi\beta n)
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
M(z;\beta) &= \frac{1}{2\pi i} \int_{Ha} e^{\sigma - z\sigma^\beta} \, \frac{d\sigma}{\sigma^{1-\beta}} \\
&= \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \, \Gamma[-\beta n + (1-\beta)]} \\
&= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!} \, \Gamma(\beta n) \, \sin(\pi\beta n)
\end{aligned}
\tag{3.16}
$$

In the theory of special functions, see Erdélyi (1955), we find an entire function, referred to as the *Wright function*, which reads (in our notation)

$$
W_{\lambda,\mu}(z) := \frac{1}{2\pi i} \int_{Ha} e^{\sigma + z\sigma^{-\lambda}} \, \frac{d\sigma}{\sigma^\mu} := \sum_{n=0}^{\infty} \frac{z^n}{n! \, \Gamma(\lambda n + \mu)}, \quad z \in \mathbb{C}, \tag{3.17}
$$

where $\lambda > -1$ and $\mu > 0$. From a comparison among (3.15-16) and (3.17) we recognize that the auxiliary functions are related to the Wright function according to

$$
F(z;\beta) = W_{-\beta,0}(-z) = \beta \, z \, M(z;\beta), \qquad M(z;\beta) = W_{-\beta,1-\beta}(-z). \tag{3.18}
$$

Although convergent in all of $\mathbb{C}$, the series representations in (3.15-16) can be used to provide a numerical evaluation of our auxiliary functions only for relatively small values of r, so that asymptotic evaluations as $r \to +\infty$ are required. Choosing as a variable r/β rather than r, the computation by the saddle-point method for the M function is easier and yields, see Mainardi and Tomirotti (1995),

$$
M(r/\beta;\beta) \sim \frac{r^{(\beta-1/2)/(1-\beta)}}{\sqrt{2\pi(1-\beta)}} \exp\left[-\frac{1-\beta}{\beta} \, r^{1/(1-\beta)}\right], \; r \to +\infty. \tag{3.19}
$$

We note that the saddle-point method for $\beta = 1/2$ provides the exact result (3.11), *i.e.* $M(r;1/2) = M^d(r) = \exp(-r^2/4)/\sqrt{\pi}$, but breaks down for $\beta \to 1^-$. The case $\beta = 1$, for which (1.2) reduces to the standard wave equation, is of course a singular limit also for the series representation since $M(r;1) = \delta(r-1)$.

The exponential decay for $r \to +\infty$ ensures that all the moments of $M(r;\beta)$ in $\mathbb{R}^+$ are finite; in particular, see Mainardi (1997), we obtain

$$\int_0^{+\infty} r^n \, M(r;\beta) \, dr = \frac{\Gamma(n+1)}{\Gamma(\beta n + 1)}, \quad n = 0, 1, 2 \ldots \qquad (3.20)$$

In Fig. 1 we exhibit plots of the auxiliary function $M(r;\beta)$ in $0 \le r \le 4$ for some rational values of β. The plots are obtained by means of a numerical matching between the series and the saddle-point representations. As a matter of fact it turns out that the function $M(r;\beta)$ is decreasing for $0 < \beta < 1/2$, while for $1/2 < \beta < 1$ it first increases and then decreases exhibiting the maximum value $M_0(\beta)$ at a certain point $r_0(\beta)$; as $\beta \to 1^-$, $M_0(\beta) \to +\infty$ and $r_0(\beta) \to 1$.
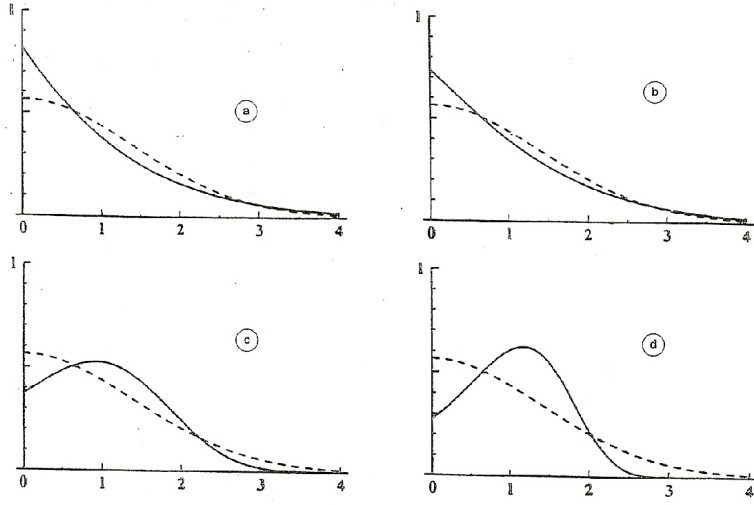

Figure 1: Comparison of $M(r;\beta)$ (continuous line) with $M(r;1/2)$ (dashed line) in $0 \le r \le 4$, for various value of β; (a) 1/4, (b) 1/3, (c) (2/3), (d) 3/4.

4 The Evolution of the Seismic Pulse from the Fundamental Solutions

It is known that in theoretical seismology the delta-Dirac function is of great relevance in simulating the pulse generated by an ideal seismic source, concentrated in space ($\delta(x)$) or in time ($\delta(t)$). Consequently, the fundamental solutions of the *Cauchy* and *Signalling* problems are those of greater interest because they provide us with information on the possible evolution of the seismic pulses during their propagation from the seismic source. Accounting of the reciprocity relation (3.12) and the similarity variable (3.13), $r = x/(\sqrt{D}\,t^\beta)$, the two fundamental solutions can be written, for $x > 0$ and $t > 0$, as

$$\mathcal{G}_c(x,t;\beta) = \frac{1}{2\beta x} \, F(r;\beta) = \frac{1}{2\sqrt{D}\,t^\beta} \, M(r;\beta), \qquad (4.1a)$$

$$\mathcal{G}_s(x,t;\beta) = \frac{1}{t} \, F(r;\beta) = \frac{\beta x}{\sqrt{D}\,t^{1+\beta}} \, M(r;\beta). \qquad (4.1b)$$

The above equations mean that for the fundamental solution of the *Cauchy* [*Signalling*] problem the time [spatial] shape is the same at each position [instant], the only changes being due to space [time] - dependent changes of width and amplitude. The maximum amplitude in time [space] varies precisely as $1/x$ [$1/t$]. The two fundamental solutions exhibit scaling properties that make easier their plots versus distance (at fixed instant) and versus time (at fixed position). In fact, using the well-known scaling properties of the Laplace transform in (3.6) and (3.8), we easily prove, for any p, $q > 0$, that

$$\mathcal{G}_c(px, qt; \beta) = \frac{1}{q^\beta}\, \mathcal{G}_c(px/q^\beta, t; \beta)\,, \qquad (4.2a)$$

$$\mathcal{G}_s(px, qt; \beta) = \frac{1}{q}\, \mathcal{G}_s(px/q^\beta, t; \beta)\,, \qquad (4.2b)$$

and, consequently, in plotting we can choose suitable values for the fixed variable.

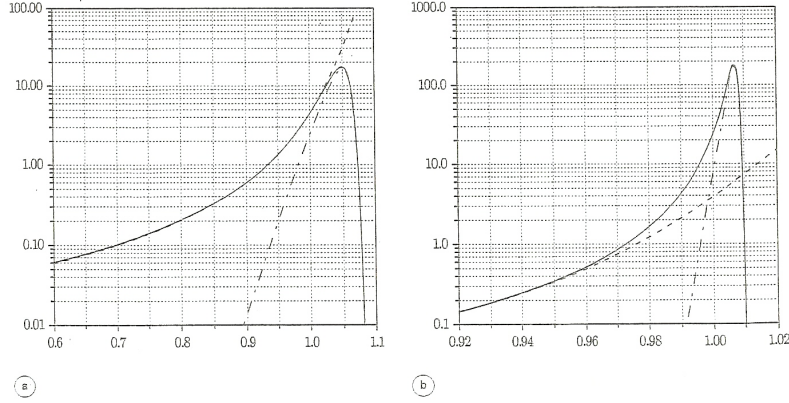Figure 2: Comparison of the representations of $M(r; \beta)$ with $\beta = 1-\epsilon$ around the maximum $r \approx 1$, in the cases (a) $\epsilon = 0.01$, (b) $\epsilon = 0.001$, obtained by Pipkin's method (continuous line), 100 terms-series (dashed line) and and saddle-point method (dashed-dotted line).

In order to inspect the evolution of the initial pulse for seismological purposes, we need to plot $\mathcal{G}_c(x, t; \beta)$ versus x and $\mathcal{G}_s(x, t; \beta)$ versus t as β is sufficiently close to 1 (*nearly elastic* cases) to ensure a sufficiently low value for the constant internal friction Q^{-1}. From (2.13)-(2.14) and (2.16)-(2.17) we need to consider $\beta = 1-\epsilon$ with $\epsilon = \nu/2$ of the order of 0.001 to 0.01. In the evaluation of the auxiliary functions in the *nearly elastic* cases, we note that the matching between the series and saddle point representations is no longer achieved since the saddle point turns out to be wide and the consequent approximation becomes poor. In these cases we need to adopt the ingenious variant of the saddle-point method introduced by Pipkin (1972-1986), see also Kreiss and Pipkin (1986), which allows us to see some structure in the peak while it tends to the Dirac delta function. With Pipkin's method we get the desired matching with the series representation just in the region around the maximum $r \approx 1$, as shown in Fig. 2a,b, where we exhibit the significant plots of the auxiliary function $M(r; \beta)$ with $\beta = 1 - \epsilon$ for $\epsilon = 0.01$ and $\epsilon = 0.001$.
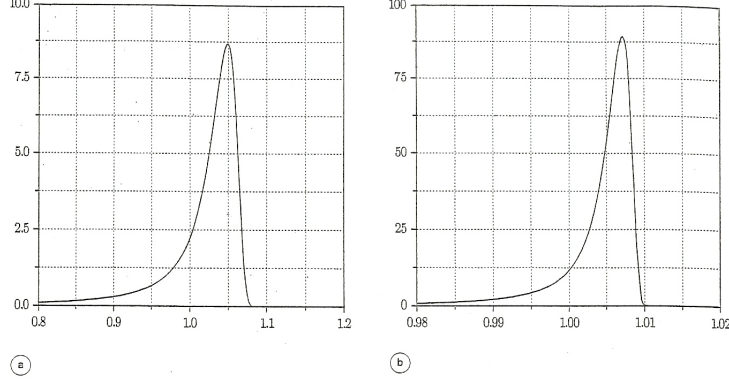
Figure 3: Plots of the fundamental solution $\mathcal{G}_c(x,t;\beta)$ versus x at fixed $t = D = 1$, with $\beta = 1 - \epsilon$ in the cases (a) $\epsilon = 0.01$, (b) $\epsilon = 0.001$.
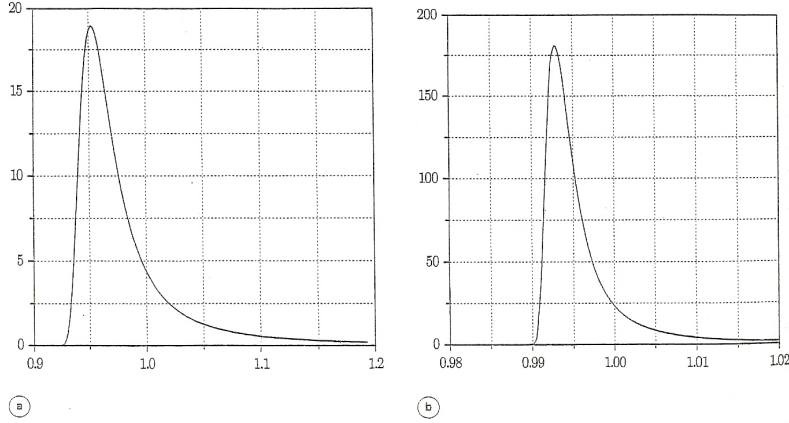

Figure 4: Plots of the fundamental solution $\mathcal{G}_s(x,t;\beta)$ versus t at fixed $x = D = 1$, with $\beta = 1 - \epsilon$ in the cases (a) $\epsilon = 0.01$, (b) $\epsilon = 0.001$.

Once obtained the auxiliary function $M(r;\beta)$ in the nearly elastic cases, we easily get the corresponding plots of the fundamental solutions of the Cauchy and Signalling problems by using (4.1a-b), see Figs. 3a,b and 4a,b.

We also note the exponential decay of $\mathcal{G}_c(x,t;\beta)$ as $x \to +\infty$ (at fixed t) and the algebraic decay of $\mathcal{G}_s(x,t;\beta)$ as $t \to +\infty$ (at fixed x), for $0 < \beta < 1$. In fact, using (4.1a,b) with (3.16) and (3.19), we get

$$\mathcal{G}_c(x,t;\beta) \sim a(t)\, x^{(\beta-1/2)/(1-\beta)} \exp\left[-b(t)x^{1/(1-\beta)}\right], \quad x \to \infty, \qquad (4.3a)$$

$$\mathcal{G}_s(x,t;\beta) \sim c(x)\, t^{-(1+\beta)}, \quad t \to \infty, \qquad (4.3b)$$

where $a(t)$, $b(t)$ and $c(x)$ are positive functions.

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5   The Fundamental Solutions as Probability Density Functions

It is well known that the fundamental solution of the standard diffusion equation for the Cauchy problem is related with the *Gauss* or *normal* probability law, bilateral in space. In fact, recalling (3.3a), we have

$$\mathcal{G}_c^d(x,t) = \frac{1}{2\sqrt{\pi\,D\,t}}\,\mathrm{e}^{-x^2/(4\,D\,t)} = p_G(x;\sigma)\,, \qquad (5.1)$$

where

$$p_G(x;\sigma) := \frac{1}{\sqrt{2\pi}\,\sigma}\,\mathrm{e}^{-x^2/(2\sigma^2)}\,,\ \sigma^2 = 2\,D\,t\,, \qquad (5.2)$$

denotes the well-known *Gauss* probability density function (*pdf*) with variance σ^2. The associated cumulative distribution function (*cdf*) is known to be

$$\begin{aligned}
\mathcal{P}_G(x;\sigma) &:= \int_{-\infty}^{x} p_G(x;\sigma)\,dx \\
&= \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x}{\sqrt{2}\,\sigma}\right)\right] = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x}{2\,\sqrt{D\,t}}\right)\right].
\end{aligned} \qquad (5.3)$$

The moments of even order of the *Gauss pdf* turn out to be with $n \in \mathbb{N}$,

$$\int_{-\infty}^{+\infty} x^{2n}\,p_G(x;\sigma)\,dx = \frac{(2n)!}{2^n\,n!}\,\sigma^{2n} = (2n-1)!!\,\sigma^{2n} = (2n-1)!!\,(2Dt)^n\,. \quad (5.4)$$

If we consider the fundamental solution of the standard diffusion equation but for the Signalling problem, we note that it is related to the *Lévy* probability law, unilateral in time (a property not so well-known as that for the Cauchy problem!). In fact, recalling (3.3b), we have

$$\mathcal{G}_s^d(x,t) = \frac{x}{2\sqrt{\pi\,D}\,t^{3/2}}\mathrm{e}^{-x^2/(4\,D\,t)} = p_L(t;\mu)\,, \qquad (5.5)$$

where

$$p_L(t;\mu) = \frac{\sqrt{\mu}}{\sqrt{2\pi}\,t^{3/2}}\,\mathrm{e}^{-\mu/(2t)}\,,\ \mu = \frac{x^2}{2\,D}\,, \qquad (5.6)$$

denotes the *Lévy pdf* , see Feller (1971), with *cdf*

$$\mathcal{P}_L(t;\mu) := \int_0^t p_L(t;\mu)\,dt = \mathrm{erfc}\left(\sqrt{\frac{\mu}{2t}}\right) = \mathrm{erfc}\left(\frac{x}{2\,\sqrt{D\,t}}\right). \qquad (5.7)$$

The *Lévy pdf* has all moments of integer order infinite, since it decays at infinity as $t^{-3/2}$. However, we note that the moments of real order δ are finite only if $0 \le \delta < 1/2$. In particular, for this *pdf* the mean (expectation) is infinite, but the *médiane* is finite. In fact, from $\mathcal{P}_L(t_{med};\mu) = 1/2$, it turns out that $t_{med} \approx 2\mu$, since the complementary error function gets the value $1/2$ as its argument is approximatively $1/2$ (a better evaluation of the argument is $1/2.1$).

The *Gauss* and *Lévy* laws are special cases of the important class of α - *stable* probability distributions, or *stable* distributions with index of stability

(or characteristic exponent) $\alpha = 2$ and $\alpha = 1/2$, respectively. Another special case is provided by the Cauchy law with *pdf* $p_C(x;\lambda) = \lambda/[\pi(x^2 + \lambda^2)]$ and $\alpha = 1$.

The name stable has been assigned to these distributions because of the following property: if two independent real random variables with the same shape or *type* of distribution are combined linearly and the distribution of the resulting random variable has the same shape, the common distribution (or its type, more precisely) is said to be *stable*. More precisely, if Y_1 and Y_2 are random variables having such distribution, then Y defined by the linear combination $cY = c_1 Y_1 + c_2 Y_2$ has a similar distribution with the same index α for any positive real values of the constants c, c_1 and c_2 with $c^\alpha = c_1^\alpha + c_2^\alpha$. As a matter of fact only the range $0 < \alpha \le 2$ is allowed for the index of stability. The case $\alpha = 2$ is noteworthy since it corresponds to the *normal* distribution, which is the only stable distribution which has finite variance, indeed finite moments of any order. In the cases $0 < \alpha < 2$ the corresponding *pdf* $p_\alpha(y)$ have inverse power tails, *i.e.* $\int_{|y|>\lambda} p_\alpha(y)\,dy = O(\lambda^{-\alpha})$ and therefore their absolute moments of order δ are finite if $0 \le \delta < \alpha$ and infinite if $\delta \ge \alpha$.

The inspiration for systematic research on stable distributions, originated with Paul Lévy, was the desire to generalize the celebrated *Central Limit Theorem (CLT)*.

The restrictive condition of stability enabled some authors to derive the general form for the characteristic function (*cf*, the Fourier transform of the *pdf*) of a stable distribution, see Feller (1971). A stable *cf* is also *infinitely divisible*, *i.e.* for every positive integer n it can be expressed as the nth power of some *cf*. Equivalently we can say that for every positive integer n a stable *pdf* can be expressed as the n-fold convolution of some *pdf*. All stable *pdf* are *unimodal* and indeed *bell-shaped*, *i.e.* their n-th derivative has exactly n zeros,

Using standardized random variables, the α-*stable* distributions turn out to depend on an additional parameter γ, said the *skewness* parameter. Denoting a stable *pdf* by $p_\alpha(y;\theta)$, we note the property $p_\alpha(-y;-\theta) = p_\alpha(y;\theta)$. Consequently a stable *pdf* with $\theta = 0$ is necessarily symmetrical. As a matter of fact $|\theta| \le \alpha$ if $0 < \alpha < 1$ and $|\theta| \le 2 - \alpha$ if $1 < \alpha < 2$. Stable distributions with extremal values of θ are called *extremal*.

From the theory one recognizes that the normal distribution is the only stable *df* independent on θ, and that all the extremal stable distributions with $0 < \alpha < 1$ are unilateral, *i.e.* vanishing in $\mathbb{R}^\pm$ if $\theta = \pm\alpha$. In particular, the following representations by convergent power series are valid for stable distributions with $0 < \alpha < 1$ (negative powers) and $1 < \alpha < 2$ (positive powers), for $y > 0$,

$$p_\alpha(y;\theta) = \frac{1}{\pi y}\sum_{n=1}^{\infty}(-y^{-\alpha})^n\,\frac{\Gamma(n\alpha+1)}{n!}\,\sin\left[\frac{n\pi}{2}(\theta-\alpha)\right]\,,\ 0 < \alpha < 1, \quad (5.8)$$

$$p_\alpha(y;\theta) = \frac{1}{\pi y}\sum_{n=1}^{\infty}(-y)^n\,\frac{\Gamma(n/\alpha+1)}{n!}\,\sin\left[\frac{n\pi}{2\alpha}(\theta-\alpha)\right]\,,\ 1 < \alpha < 2. \quad (5.9)$$

From (5.8)-(5.9) a relation between stable *pdf* with index α and $1/\alpha$ can be derived. Assuming $1/2 < \alpha < 1$ and $y > 0$, we obtain

$$\frac{1}{y^{\alpha+1}}\,p_{1/\alpha}(y^{-\alpha};\theta) = p_\alpha(y;\theta^*)\,,\ \theta^* = \alpha(\theta+1) - 1\,. \quad (5.10)$$

A quick check shows that θ^* falls within the prescribed range, $|\theta^*| \leq \alpha$, provided that $|\theta| \leq 2 - 1/\alpha$. Furthermore, we can derive a relation between extremal stable *pdf* and our auxiliary functions of Wright type. In fact, by comparing (5.8-9) with the series representations in (3.15-16) and using (3.18), we obtain

$$p_\alpha(y; -\alpha) = \frac{1}{y} F(y^{-\alpha}; \alpha) = \frac{\alpha}{y^{\alpha+1}} M(y^{-\alpha}; \alpha), \ 0 < \alpha < 1, \qquad (5.11)$$

$$p_\alpha(y; \alpha - 2) = \frac{1}{y} F(y; 1/\alpha) = \frac{1}{\alpha} M(y; 1/\alpha), \ 1 < \alpha < 2. \qquad (5.12)$$

Consequently we can interpret the fundamental solutions (4.1a) and (4.1b) in terms of stable *pdf*, so generalizing the arguments for the standard diffusion equation based on (5.1-7).

We easily recognize that for $0 < \beta < 1$ the fundamental solution for the Signalling problem provides a unilateral extremal stable *pdf* in (scaled) time with index of stability $\alpha = \beta$, which decays according to (4.3b) with a power law. In fact, from (4.1b) and (5.11) we note that, putting $y = r^{-1/\beta} = \tau$,

$$(x/\sqrt{D})^{1/\beta} \, \mathcal{G}_s(x, t; \beta) = p_\beta(\tau; -\beta), \quad \tau = t \, (\sqrt{D}/x)^{1/\beta} > 0. \qquad (5.13)$$

This property has been noted also by Kreiss and Pipkin (1986) based on (3.8) and on Feller's result, $p_\alpha(t; -\alpha) \div \exp(-s^\alpha)$ for $0 < \alpha < 1$.

As far as the Cauchy problem is concerned, we note that the corresponding fundamental solution provides a bilateral symmetrical *pdf* in (scaled) distance with two branches, for $x > 0$ and $x < 0$, obtained one from the other by reflection. For large $|x|$ each branch exhibits an exponential decay according to (4.3) and, only for $1/2 \leq \beta < 1$, it is the corresponding branch of an extremal stable *pdf* with index of stability $\alpha = 1/\beta$. In fact, from (4.1b) and (5.12) we note that, putting $y = |r| = \xi > 0$,

$$2\beta \sqrt{D} \, t^\beta \, \mathcal{G}_c(|x|, t; \beta) = p_{1/\beta}(\xi, 1/\beta - 2), \quad \xi = |x|/(\sqrt{D} \, t^\beta) > 0. \qquad (5.14)$$

This property had to the authors' knowledge not been noted: it properly generalizes the Gaussian property of the *pdf* found for $\beta = 1/2$ (standard diffusion). Furthermore, using (3.20), the moments (of even order) of $\mathcal{G}_c(x, t; \beta)$ turn out to be

$$\int_{-\infty}^{+\infty} x^{2n} \, \mathcal{G}_c(x, t; \beta) \, dx = \frac{\Gamma(2n + 1)}{\Gamma(2\beta n + 1)} \, (Dt^{2\beta})^n, \ n = 0, \, 1, \, 2, \, \ldots \qquad (5.15)$$

We recognize that the variance is now proportional to $Dt^{2\beta}$, which implies a phenomenon of *fast* diffusion if $1/2 < \beta < 1$.

Appendix: Essentials of Fractional Calculus

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order. The term *fractional* is a misnomer, but it is retained following the prevailing use.

According to the Riemann-Liouville approach to fractional calculus, the notion of fractional integral of order α ($\alpha > 0$) is a natural consequence of the well known formula (usually attributed to Cauchy), that reduces the calculation of the $n-$fold primitive of a function $f(t)$ to a single integral of convolution type. In our notation the Cauchy formula reads

$$J^n f(t) := f_n(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau)\, d\tau\,, \quad t > 0\,, \quad n \in \mathbb{N}\,, \quad (A.1)$$

where $\mathbb{N}$ is the set of positive integers. From this definition we note that $f_n(t)$ vanishes at $t = 0$ with its derivatives of order $1, 2, \ldots, n-1$. For convention we require that $f(t)$ and henceforth $f_n(t)$ be a *causal* function, *i.e.* identically vanishing for $t < 0$.

In a natural way one is led to extend the above formula from positive integer values of the index to any positive real values by using the Gamma function. Indeed, noting that $(n-1)! = \Gamma(n)$, and introducing the arbitrary *positive* real number α, one defines the _Fractional Integral of order $\alpha > 0$_ :

$$J^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau\,, \quad t > 0\,, \quad \alpha \in \mathbb{R}^+\,, \qquad (A.2)$$

where $\mathbb{R}^+$ is the set of positive real numbers. For complementation we define $J^0 := I$ (Identity operator), *i.e.* we mean $J^0 f(t) = f(t)$. Furthermore, by $J^\alpha f(0^+)$ we mean the limit (if it exists) of $J^\alpha f(t)$ for $t \to 0^+$; this limit may be infinite.

We note the *semigroup property* $J^\alpha J^\beta = J^{\alpha+\beta}$, α, $\beta \geq 0$, which implies the *commutative property* $J^\beta J^\alpha = J^\alpha J^\beta$, and the effect of our operators J^α on the power functions

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}\, t^{\gamma+\alpha}\,, \quad \alpha \geq 0\,, \quad \gamma > -1\,, \quad t > 0\,. \qquad (A.3)$$

These properties are of course a natural generalization of those known when the order is a positive integer.

Introducing the Laplace transform by the notation $\mathcal{L}\{f(t)\} := \int_0^\infty e^{-st} f(t)\, dt = \widetilde{f}(s)$, $s \in \mathbb{C}$, and using the sign $\div$ to denote a Laplace transform pair, *i.e.* $f(t) \div \widetilde{f}(s)$, we note the following rule for the Laplace transform of the fractional integral,

$$J^\alpha f(t) \div \frac{\widetilde{f}(s)}{s^\alpha}\,, \quad \alpha \geq 0\,, \qquad (A.4)$$

which is the generalization of the case with an n-fold repeated integral.

After the notion of fractional integral, that of fractional derivative of order α ($\alpha > 0$) becomes a natural requirement and one is attempted to substitute α with $-\alpha$ in the above formulas. However, this generalization needs some care in order to guarantee the convergence of the integrals and preserve the well known properties of the ordinary derivative of integer order.

Denoting by D^n with $n \in \mathbb{N}$, the operator of the derivative of order n, we first note that $D^n J^n = I$, $J^n D^n \neq I$, $n \in \mathbb{N}$, *i.e.* D^n is left-inverse (and

not right-inverse) to the corresponding integral operator J^n. In fact we easily recognize from (A.1) that

$$J^n \, D^n \, f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \, \frac{t^k}{k!} \, , \quad t > 0 \, . \qquad (A.5)$$

As a consequence we expect that D^α is defined as left-inverse to J^α. For this purpose, introducing the positive integer m such that $m - 1 < \alpha \le m$, one defines the *Fractional Derivative of order $\alpha > 0$* as $D^\alpha f(t) := D^m \, J^{m-\alpha} f(t)$, i.e.

$$D^\alpha f(t) := \begin{cases} \dfrac{d^m}{dt^m} \left[\dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \dfrac{f(\tau)}{(t-\tau)^{\alpha+1-m}} \, d\tau \right], & m - 1 < \alpha < m, \\[12pt] \dfrac{d^m}{dt^m} f(t), & \alpha = m \, . \end{cases} \qquad (A.6)$$

Defining for complementation $D^0 = J^0 = I$, then we easily recognize that $D^\alpha \, J^\alpha = I$, $\alpha \ge 0$, and

$$D^\alpha \, t^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 - \alpha)} \, t^{\gamma-\alpha} \, , \quad \alpha \ge 0 \, , \quad \gamma > -1 \, , \quad t > 0 \, . \qquad (A.7)$$

Of course, these properties are a natural generalization of those known when the order is a positive integer.

Note the remarkable fact that the fractional derivative $D^\alpha f$ is not zero for the constant function $f(t) \equiv 1$ if $\alpha \notin \mathbb{N}$. In fact, (A.7) with $\gamma = 0$ teaches us that

$$D^\alpha 1 = \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} \, , \quad \alpha \ge 0 \, , \quad t > 0 \, . \qquad (A.8)$$

This, of course, is $\equiv 0$ for $\alpha \in \mathbb{N}$, due to the poles of the gamma function in the points $0, -1, -2, \dots$. We now observe that an alternative definition of fractional derivative, originally introduced by Caputo (1967) (1969) in the late sixties and adopted by Caputo and Mainardi (1971) in the framework of the theory of *Linear Viscoelasticity*, is $D_*^\alpha f(t) := J^{m-\alpha} \, D^m f(t)$ with $m - 1 < \alpha \le m$, $m \in \mathbb{N}$, i.e.

$$D_*^\alpha f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \dfrac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} \, d\tau \, , & m - 1 < \alpha < m \, , \\[12pt] \dfrac{d^m}{dt^m} f(t), & \alpha = m \, . \end{cases} \qquad (A.9)$$

This definition is of course more restrictive than (A.6), in that requires the absolute integrability of the derivative of order m. Whenever we use the operator D_*^α we (tacitly) assume that this condition is met. We easily recognize that in general

$$D^\alpha f(t) := D^m \, J^{m-\alpha} f(t) \ne J^{m-\alpha} \, D^m f(t) := D_*^\alpha f(t) \, , \qquad (A.10)$$

unless the function $f(t)$ along with its first $m - 1$ derivatives vanishes at $t = 0^+$. In fact, assuming that the passage of the m-derivative under the integral is legitimate, one recognizes that, for $m - 1 < \alpha < m$ and $t > 0$,

$$D^\alpha f(t) = D_*^\alpha f(t) + \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k - \alpha + 1)} \, f^{(k)}(0^+) \, , \qquad (A.11)$$

and therefore, recalling the fractional derivative of the power functions (A.7),

$$D^\alpha \left(f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^{(k)}(0^+) \right) = D_*^\alpha f(t) \,. \qquad (A.12)$$

The alternative definition (A.9) for the fractional derivative thus incorporates the initial values of the function and of its integer derivatives of lower order. The subtraction of the Taylor polynomial of degree $m-1$ at $t = 0^+$ from $f(t)$ means a sort of regularization of the fractional derivative. In particular, according to this definition, the relevant property for which the fractional derivative of a constant is still zero can be easily recognized, *i.e.*

$$D_*^\alpha 1 \equiv 0 \,, \quad \alpha > 0 \,. \qquad (A.13)$$

We now explore the most relevant differences between the two fractional derivatives (A.6) and (A.9). We agree to denote (A.9) as the *Caputo fractional derivative* to distinguish it from the standard Riemann-Liouville fractional derivative (A.6). We observe, again by looking at (A.7), that $D^\alpha t^{\alpha-1} \equiv 0$, $\alpha > 0$, $t > 0$. From above we thus recognize the following statements about functions which for $t > 0$ admit the same fractional derivative of order α, with $m - 1 < \alpha \le m$, $m \in \mathbb{N}$,

$$D^\alpha f(t) = D^\alpha g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j \, t^{\alpha-j} \,, \qquad (A.14)$$

$$D_*^\alpha f(t) = D_*^\alpha g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j \, t^{m-j} \,. \qquad (A.15)$$

In these formulas the coefficients c_j are arbitrary constants.

For the two definitions we also note a difference with respect to the *formal* limit as $\alpha \to (m-1)^+$. From (A.6) and (A.9) we obtain respectively,

$$\alpha \to (m-1)^+ \implies D^\alpha f(t) \to D^m J f(t) = D^{m-1} f(t) \,; \qquad (A.16)$$

$$\alpha \to (m-1)^+ \implies D_*^\alpha f(t) \to J D^m f(t) = D^{m-1} f(t) - f^{(m-1)}(0^+) \,. \quad (A.17)$$

We now consider the *Laplace transform* of the two fractional derivatives. For the standard fractional derivative D^α the Laplace transform, assumed to exist, requires the knowledge of the (bounded) initial values of the fractional integral $J^{m-\alpha}$ and of its integer derivatives of order $k = 1, 2, \ldots, m-1$. The corresponding rule reads, in our notation,

$$D^\alpha f(t) \div s^\alpha \, \widetilde{f}(s) - \sum_{k=0}^{m-1} D^k \, J^{(m-\alpha)} f(0^+) \, s^{m-1-k} \,, \quad m - 1 < \alpha \le m \,. \quad (A.18)$$

The *Caputo fractional derivative* appears more suitable to be treated by the Laplace transform technique in that it requires the knowledge of the

(bounded) initial values of the function and of its integer derivatives of order $k = 1, 2, \ldots, m - 1$, in analogy with the case when $\alpha = m$. In fact, by using (A.4) and noting that

$$J^\alpha \, D_*^\alpha \, f(t) = J^\alpha \, J^{m-\alpha} \, D^m \, f(t) = J^m \, D^m \, f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!}. \quad (A.19)$$

we easily prove the following rule for the Laplace transform,

$$D_*^\alpha \, f(t) \div s^\alpha \, \widetilde{f}(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \, s^{\alpha-1-k}, \quad m - 1 < \alpha \le m. \quad (A.20)$$

Indeed, the result (A.20), first stated by Caputo (1969) by using the Fubini-Tonelli theorem, appears as the most "natural" generalization of the corresponding result well known for $\alpha = m$.

This appendix is based on the review by Gorenflo and Mainardi (1997). For more details on the classical treatment of fractional calculus the reader is referred to Erdélyi (1954), Oldham and Spanier (1974), Samko *et al.* (1987-1993) and Miller and Ross (1993). Gorenflo and Mainardi have pointed out the major utility of the Caputo fractional derivative in the treatment of differential equations of fractional order for *physical applications.* In fact, in physical problems, the initial conditions are usually expressed in terms of a given number of bounded values assumed by the field variable and its derivatives of integer order, no matter if the governing evolution equation may be a generic integro-differential equation and therefore, in particular, a fractional differential equation.

References

Aki, K. and P.G. Richards (1980): *Quantitative Seismology* (Freeman, San Francisco), Vol. 1, Ch. 5, pp. 167-185.

Ben-Menahem, A., and S.J. Singh (1981): *Seismic Waves and Sources* (Springer-Verlag, New York), Ch. 10, pp. 840-944.

Buchen, P.W. and F. Mainardi (1975): Asymptotic expansions for transient viscoelastic waves, *J. Méc.* **14**, 597-608.

Caputo, M. (1966) : Linear models of dissipation whose Q is almost frequency independent, *Annali di Geofisica*, **19**, 383-393.

Caputo, M. (1967) : Linear models of dissipation whose Q is almost frequency independent, Part II., Geophys. J. R. Astr. Soc., **13**, 529-539.

Caputo, M. (1969): *Elasticità e Dissipazione* (Zanichelli Bologna). [in Italian]

Caputo, M. and F. Mainardi (1971): Linear models of dissipation in anelastic solids, *Riv. Nuovo Cimento* (Ser II) **1**, 161-198.

Caputo, M. (1976): Vibrations of an infinite plate with a frequency independent Q, *J. Acoust. Soc. Am.*, **60**, 634-639.

Caputo, M. (1979): A model for the fatigue in elastic materials with frequency independent Q, *J. Acoust. Soc. Am.*, **66**, 176-179.

Caputo, M. (1981): Elastic radiation from a source in a medium with an almost frequency independent Q, *J. Phys. Earth*, **29**, 487-497.

Caputo, M. (1985) : Generalized rheology and geophysical consequences, *Tectonophysics*, **116**, 163-172.

Caputo, M. (1996)a: Modern rheology and electric induction: multivalued index of refraction, splitting of eigenvalues and fatigues, *Annali di Geofisica*, **39**, 941-966.

Caputo, M. (1996)b: The Green function of the diffusion in porous media with memory, *Rend. Fis. Acc. Lincei* (Ser. 9), **7**, 243-250.

Carcione, J.M., Kosloff, D. and R. Kosloff (1988): Wave propagation in a linear viscoelastic medium, *Geophys. J.*, **95**, 597-611.

Chin, R.C.Y. (1980): Wave propagation in viscoelastic media, in *Physics of the Earth's Interior*, edited by A. Dziewonski and E. Boschi (North-Holland, Amsterdam), pp. 213-246. [E. Fermi Int. School, Course 78]

Christensen, R.M. (1982): *Theory of Viscoelasticity* (Academic Press, New York). [1-st ed. (1972)]

Erdélyi, A. *Editor* (1954): *Tables of Integral Transforms*, Bateman Project (McGraw-Hill, New York), Vol. 2, Ch. 13, pp. 181-212.

Erdélyi, A. *Editor* (1955): *Higher Transcendental Functions*, Bateman Project (McGraw-Hill, New York), Vol. 3, Ch. 18, pp. 206-227.

Feller, W. (1971), *An Introduction to Probability Theory and its Applications*, (Wiley, New York), Vol. II, Ch. 6: pp. 169-176, Ch. 13: pp. 448-454. [1-st ed. (1966)]

Futterman, W.I. (1962): Dispersive Body Waves, *J. Geophys. Res.*, **67**, 5279-5291.

Gel'fand, I.M. and G.E. Shilov (1964): *Generalized Functions*, (Academic Press, New York), Vol. I.

Giona, M. and H.E. Roman (1992): Fractional diffusion equation for transport phenomena in random media, *Physica A*, **185**, 82-97.

Gordon, R.B. and C.W. Nelson (1966): Anelastic properties of the Earth, *Rev. Geophys.*, **4**, 457-474 (1966).

Gorenflo, R. and F. Mainardi (1997): Fractional calculus: integral and differential equations of fractional order, in *Fractals and Fractional Calculus in Continuum Mechanics*, edited by A. Carpinteri and F. Mainardi (Springer Verlag, Wien), 223-276.

Graffi, D. (1982): Mathematical models and waves in linear viscoelasticity, in *Wave Propagation in Viscoelastic Media*, edited by F. Mainardi (Pitman, London), pp. 1-27. [Res. Notes in Maths, Vol. 52]

Hunter, S.C. (1960): Viscoelastic Waves, in *Progress in Solid Mechanics*, edited by I. Sneddon and R. Hill (North-Holland, Amsterdam), Vol 1, pp. 3-60.

Jackson, D.D. and D.L. Anderson (1970): Physical mechanisms for seismic wave attenuation, *Rev. Geophys.*, **2**, 625-660 (1964).

Kanamori, H. and D.L. Anderson (1977): Importance of physical dispersion in surface wave and free oscillation problems, *Rev. Geophys.*, **15**, 105-112 (1977).

Kang, I.B. and G.A. McMechan (1993): Effects of viscoelasticity on wave propagation in fault zones, near-surfaces sediments and inclusions, *Bull. Seism. Soc. Am.*, **83**, 890-906.

Kjartansson, E. (1979): Constant-Q wave propagation and attenuation, *J. Geophys. Res.*, **94**, 4737-4748.

Knopoff, L. (1964): Q, *Rev. Geophys.*, **2**, 625-660.

Kolsky, H. (1956): The propagation of stress pulses in viscoelastic solids, *Phil. Mag.* (Ser 8), **2**, 693-710.

Körnig, M. and G. Müller (1989): Rheological models and interpretation of postglacial uplift, *Geophys. J. Int.*, **98**, 243-253.

Kreis, A. and A.C. Pipkin (1986): Viscoelastic pulse propagation and stable probability distributions, *Quart. Appl. Math.*, **44**, 353-360.

Mainardi, F. and G. Turchetti (1975): Wave front expansion for transient viscoelastic waves, *Mech. Res. Comm.* **2**, 107-112.

Mainardi, F. (1994): On the initial value problem for the fractional diffusion-wave equation, in *Waves and Stability in Continuous Media* edited by S. Rionero and T. Ruggeri, (World Scientific, Singapore), pp. 246-251.

Mainardi, F. (1995): Fractional diffusive waves in viscoelastic solids in *IUTAM Symposium - Nonlinear Waves in Solids*, edited by J. L. Wegner and F. R. Norwood (ASME/AMR, Fairfield NJ), pp. 93-97. [Abstract in *Appl. Mech. Rev.*, **46** (1993), 549]

Mainardi, F. and M. Tomirotti (1995): On a special function arising in the time fractional diffusion-wave equation, in *Transform Methods and Special Functions, Sofia 1994*, edited by P. Rusev, I. Dimovski and V. Kiryakova, (Science Culture Technology, Singapore), pp. 171-183.

Mainardi, F. (1996)a: Fractional relaxation-oscillation and fractional diffusion-wave phenomena, *Chaos, Solitons & Fractals*, **7**, 1461-1477.

Mainardi, F. (1996)b: The fundamental solutions for the fractional diffusion-wave equation, *Applied Mathematics Letters*, **9**, No 6, 23-28.

Mainardi, F. (1997): Fractional calculus; some basic problems in continuum and statistical mechanics, in *Fractals and Fractional Calculus in Continuum Mechanics*, edited by. A. Carpinteri and F. Mainardi (Springer-Verlag, Wien), 291-348.

Meshkov, S.I. and Yu. A. Rossikhin (1970): Sound wave propagation in a viscoelastic medium whose hereditary properties are determined by weakly singular kernels, in *Waves in Inelastic Media*, edited by Yu. N. Rabotnov (Kishniev), pp. 162-172. [in Russian]

Metzler, R., Glöckle, W.G. and T.F. Nonnenmacher (1994): Fractional model equation for anomalous diffusion, *Physica A*, **211**, 13-24.

Miller, K.S. and B. Ross (1993): *An Introduction to the Fractional Calculus and Fractional Differential Equations* (Wiley, New York).

Mitchell, B.J. (1995): Anelastic structure and evolution of the continental crust and upper mantle from seismic surface wave attenuation, *Rev. Geophys.*, **33**, 441-462 (1995).

Murphy, W.F. (1982): Effect of partial water saturation on attenuation in sandstones, *J. Acoust. Soc. Am.*, **71**, 1458-1468.

Nigmatullin, R.R. (1986): The realization of the generalized transfer equation in a medium with fractal geometry, *Phys. Stat. Sol. B*, **133**, 425-430. [English transl. from Russian]

O'Connell, R.J. and B. Budiansky (1978): Measures of dissipation in viscoelastic media, *Geophys. Res. Lett.*, **5**, 5-8.

Oldham, K.B. and J. Spanier (1974): *The Fractional Calculus* (Academic Press, New York).

Pipkin, A.C. (1986): *Lectures on Viscoelastic Theory* (Springer-Verlag, New York), Ch. 4, pp. 56-76. [1-st ed, 1972]

Ranalli, G. (1987): *Rheology of the Earth* (Allen & Unwin, London).

Rossikhin, Yu. A. and M.V. Shitikova (1997): Application of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids, *Appl. Mech. Rev*, **50**, 15-67.

Sabadini, R., Yuen, D.A. and P. Gasperini (1985): The effects of transient rheology on the interpretation of lower mantle viscosity, *Geophys. Res. Lett.*, **12**, 361-364.

Sabadini, R., Smith, B.K. and D.A. Yuen (1987): Consequences of experimental transient rheology, *Geophys. Res. Lett.*, **14**, 816-819.

Samko S.G., Kilbas, A.A. and O.I. Marichev (1993): *Fractional Integrals and Derivatives, Theory and Applications*, (Gordon and Breach, Amsterdam). [Engl. Transl. from Russian, *Integrals and Derivatives of Fractional Order and Some of Their Applications*, Nauka i Tekhnika, Minsk (1987)]

Savage, J.C. and M.E. O'Neill (1975): The relation between the Lomnitz and the Futterman theories of internal friction, *J. Geophys. Res.*, **80**, 249-251.

Schneider, W.R. and W. Wyss (1989): Fractional diffusion and wave equations, *J. Math. Phys.*, **30**, 134-144.

Spencer , J. W. (1981): Stress relaxation at low frequencies in fluid saturated rocks; attenuation and modulus dispersion, *J. Geophys. Res.*, **86**, 1803-1812.

Strick, E. (1967): The determination of Q, dynamic viscosity and creep curves from wave propagation measurements, *Geophys. J. R. Astr. Soc.*, **13**, 197-218.

Strick, E. (1970): A predicted pedestal effect for pulse propagation in constant-Q solids, *Geophysics*, **35**, 387-403.

Strick, E. (1982): Application of linear viscoelasticity to seismic wave propagation, in *Wave Propagation in Viscoelastic Media*, edited by F. Mainardi (Pitman, London), pp. 169-193. [Res. Notes in Maths, Vol. 52]

Strick, E. and F. Mainardi (1982): On a general class of constant Q solids, *Geophys. J. R. Astr. Soc.*, **69**, 415-429.

Strick, E. (1984): Implication of Jeffreys-Lomnitz transient creep, *J. Geophys. Res.*, **89**, 437-451.

Yuen, D.A., Sabadini, R., Gasperini, P. and E. Boschi (1986): On transient rheology and glacial isostasy, *J. Geophys. Res.*, **91**, 11420-11438.